

WRONSKIAN AND DEEP ZEROS OF HOLOMORPHIC FUNCTIONS

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ABSTRACT. Given linearly independent holomorphic functions f_0, \dots, f_n on a planar domain Ω , let \mathcal{E} be the set of those points $z \in \Omega$ where a nontrivial linear combination $\sum_{j=0}^n \lambda_j f_j$ may have a zero of multiplicity greater than n , once the coefficients $\lambda_j = \lambda_j(z)$ are chosen appropriately. An elementary argument involving the Wronskian W of the f_j 's shows that \mathcal{E} is a discrete subset of Ω (and is actually the zero set of W); thus “deep” zeros are rare. We elaborate on this by studying similar phenomena in various function spaces on the unit disk, with more sophisticated boundary smallness conditions playing the role of deep zeros.

Résumé. Etant données des fonctions holomorphes f_0, \dots, f_n linéairement indépendantes sur un domaine Ω du plan, soit \mathcal{E} l'ensemble des points $z \in \Omega$ où une combinaison linéaire non triviale $\sum_{j=0}^n \lambda_j f_j$ peut avoir un zéro d'ordre supérieur à n . Un argument élémentaire utilisant le wronskien des f_j montre que \mathcal{E} est un sous-ensemble discret de Ω ; ainsi, les zéros “profonds” sont rares. Nous étudions des phénomènes similaires dans divers espaces de fonctions sur le disque unité, avec des conditions plus sophistiquées de décroissance au bord à la place de zéros profonds intérieurs.

1. INTRODUCTION

Let Ω be a domain in the complex plane \mathbb{C} , and let $\mathcal{H}(\Omega)$ denote the set of all holomorphic functions on Ω . The classical uniqueness theorem tells us that, given a non-null function $f \in \mathcal{H}(\Omega)$, the zero set $\mathcal{Z}(f) := \{z \in \Omega : f(z) = 0\}$ is discrete in Ω (i. e., has no accumulation points therein). Less known is the fact that this admits a natural extension to linear combinations of several functions, provided that the zeros are required to have suitably high multiplicities. Before stating the result, we need to introduce a bit of terminology. Namely, given a nonnegative integer n , a function $f \in \mathcal{H}(\Omega)$ and a point $z_0 \in \mathcal{Z}(f)$, we say that the zero z_0 is *n-deep* for f if its multiplicity is at least $n + 1$.

Theorem A. *Suppose f_0, \dots, f_n are linearly independent holomorphic functions on a domain $\Omega \subset \mathbb{C}$. Then there is a discrete subset \mathcal{E} of Ω with the following property: whenever $\lambda_0, \dots, \lambda_n$ are complex numbers with $\sum_{j=0}^n |\lambda_j| > 0$, the n -deep zeros of the function $\sum_{j=0}^n \lambda_j f_j$ are all contained in \mathcal{E} .*

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Thus, n -deep zeros are forbidden for a nontrivial linear combination $\sum_{j=0}^n \lambda_j f_j$ except on a “thin” set, which depends only on the f_j ’s but not on the λ_j ’s. Of course, this is no longer true with “ $(n-1)$ -deep” in place of “ n -deep”; to see why, consider the polynomials $(z-a)^n$ with $a \in \Omega$.

We strongly believe that Theorem A should be known. However, having found no reference for it in the literature, we give a simple proof instead.

Proof of Theorem A. For a point $z \in \Omega$ to be an n -deep zero of $g := \sum_{j=0}^n \lambda_j f_j$, it is necessary and sufficient that

$$g(z) = g'(z) = \cdots = g^{(n)}(z) = 0.$$

We now rewrite this as

$$(1.1) \quad \sum_{j=0}^n \lambda_j f_j^{(k)}(z) = 0 \quad (k = 0, \dots, n)$$

and view (1.1) as a system of homogeneous linear equations with “unknowns” λ_j . A nontrivial solution $(\lambda_0, \dots, \lambda_n)$ to (1.1) will therefore exist if and only if the coefficient matrix

$$\left\{ f_j^{(k)}(z) : j, k = 0, \dots, n \right\}$$

is singular. In other words, the *Wronskian* $W = W(f_0, \dots, f_n)$ defined by

$$(1.2) \quad W(f_0, \dots, f_n) := \begin{vmatrix} f_0 & f_1 & \cdots & f_n \\ f_0' & f_1' & \cdots & f_n' \\ \cdots & \cdots & \cdots & \cdots \\ f_0^{(n)} & f_1^{(n)} & \cdots & f_n^{(n)} \end{vmatrix}$$

must vanish at z . (We mention in passing that, according to some authors, the credit for introducing determinants of the form (1.2) should definitely be shared by Wronski with Froufrou.) The f_j ’s being linearly independent, it follows that $W \not\equiv 0$; see [25, Chapter 1]. Of course, it is also true that $W \in \mathcal{H}(\Omega)$, so the zero set $\mathcal{Z}(W) =: \mathcal{E}$ is a discrete subset of Ω . On the other hand, we have just seen that \mathcal{E} consists of precisely those points in Ω which can be realized as n -deep zeros for nontrivial linear combinations of the f_j ’s. \square

The proof tells us that the n -deep zeros of *all* the linear combinations as above coincide with the zeros of a *single* holomorphic function, namely, of W . In some special cases, one is able to compute W explicitly and then to determine the exceptional set $\mathcal{E} = \mathcal{Z}(W)$. We take the liberty of including one such result, which concerns the zeros of a “fewnomial” (i.e., a possibly lacunary polynomial) of the form

$$(1.3) \quad P(z) = \sum_{j=0}^n a_j z^{d_j},$$

as well as those of an exponential sum

$$(1.4) \quad Q(z) = \sum_{j=0}^n a_j e^{\mu_j z}.$$

Corollary 1.1. *Let a_0, \dots, a_n be complex numbers with $\sum_{j=0}^n |a_j| > 0$.*

(a) *Given nonnegative integers $d_0 < d_1 < \dots < d_n$, the polynomial P defined by (1.3) has no n -deep zeros in $\mathbb{C} \setminus \{0\}$.*

(b) *Given pairwise distinct complex numbers μ_0, \dots, μ_n , the function Q defined by (1.4) has no n -deep zeros in \mathbb{C} .*

To prove (a), one notes that the Wronskian $W = W(z^{d_0}, \dots, z^{d_n})$ is a monomial. Indeed, it equals cz^d with suitable integers $c \neq 0$ and $d \geq 0$ depending on the d_j 's and on n (a precise formula can be found in [2]). Thus W has no zeros in $\mathbb{C} \setminus \{0\}$, and accordingly, P has no n -deep zeros except possibly at 0.

To prove (b), one checks that the Wronskian $W(e^{\mu_0 z}, \dots, e^{\mu_n z})$ is a constant multiple of $e^{\mu z}$, where $\mu = \sum_{j=0}^n \mu_j$, the constant factor being nonzero. This time, we see that the Wronskian is nowhere zero, and the required fact follows.

One may observe that part (a) is actually a consequence of (b), and anyhow, both statements are probably – if not certainly – known. Nevertheless, just as with Theorem A above, we have found it easier to give a quick proof than to search for a reference.

While Theorem A is essentially “algebraic” in nature, we are interested in extending it to a more “analytic” context. In what follows, the domain Ω is (almost always) taken to be the disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, the functions f_0, \dots, f_n are assumed to lie in a certain space $X \subset \mathcal{H}(\mathbb{D})$, and the smallness condition imposed on the linear combination $\sum_{j=0}^n \lambda_j f_j$ involves some sort of decay near (some parts of) the circle $\mathbb{T} := \partial\mathbb{D}$ rather than having deep zeros inside. Once the class X and the decay condition are chosen appropriately (the latter being sufficiently strong), the phenomenon underlying Theorem A will manifest itself in some form or other, and we find various instances of this.

Typically, the union of the smallness sets that correspond to *all* the nontrivial linear combinations of f_0, \dots, f_n ($\in X$) turns out to be “thin”, and can be realized as a set on which a *single* nontrivial function (from a certain space $\tilde{X} \subset \mathcal{H}(\mathbb{D})$ related to X) is small, possibly not in the original sense. This general principle does not seem to have been either noticed or reflected in the literature, so we wish to highlight it here. Furthermore, a number of concrete quantitative statements will be supplied to illustrate it. Wronskians and their basic properties will again play an appreciable role in the proofs, but more sophisticated tools from complex analysis will also be needed.

In Section 2 below, we deal with “large analytic functions” on \mathbb{D} (for which a certain controlled growth near \mathbb{T} is allowed) and study their nontangential decay near \mathbb{T} . In Sections 3 and 4, we turn to spaces of “smooth analytic functions” (this time, a boundary smoothness condition is imposed) and look at the inner factors of such functions. Finally, in Section 5, we consider several types of holomorphic spaces X and discuss the exceptional sets \mathcal{E} that arise in Theorem A when the functions f_j range over X .

In conclusion, we mention that this paper shares some common features with the author’s recent work in [17, 18, 19], where Wronskians were employed in connection

with function-theoretic analogues of the so-called *abc* conjecture. Also, a portion of our current Section 4 was previously announced in [20], in rather a sketchy form.

2. LARGE ANALYTIC FUNCTIONS THAT ARE SMALL NEAR THE BOUNDARY

This section is devoted to the *Korenblum classes* $A^{-\beta}$ with $\beta > 0$; here $A^{-\beta}$ is defined as the set of all functions $f \in \mathcal{H}(\mathbb{D})$ that satisfy

$$(2.1) \quad \sup_{z \in \mathbb{D}} |f(z)|(1 - |z|)^\beta < \infty.$$

A discussion of these spaces, as well as of $A^{-\infty} := \bigcup_{\beta > 0} A^{-\beta}$, can be found in [24].

More generally, given a number $\alpha \in \mathbb{R}$, we denote by A^α the set of those $f \in \mathcal{H}(\mathbb{D})$ for which

$$(2.2) \quad \sup_{z \in \mathbb{D}} |f^{(m)}(z)|(1 - |z|)^{m-\alpha} < \infty,$$

where m is the least nonnegative integer in the interval (α, ∞) . When $\alpha = -\beta < 0$, one takes $m = 0$ and recovers the growth condition (2.1). When $\alpha > 0$, (2.2) becomes a smoothness condition on \mathbb{T} that characterizes the classical *Lipschitz–Zygmund spaces* (to be dealt with in the next section). Finally, the value $\alpha = 0$ corresponds to the *Bloch space* A^0 , which is usually denoted by \mathcal{B} ; see [1].

Now, for a point $\zeta \in \mathbb{T}$ and a number $M > 1$, we write

$$\Gamma_M(\zeta) := \{z \in \mathbb{D} : |\zeta - z| \leq M(1 - |z|)\};$$

thus $\Gamma_M(\zeta)$ is a *Stolz angle* (or *cone*) with vertex ζ . Further, given a number $\gamma > 0$, we say that a function $f \in \mathcal{H}(\mathbb{D})$ is *nontangentially small of order γ at ζ* if

$$f(z) = O((1 - |z|)^\gamma) \quad \text{as } |z| \rightarrow 1, \ z \in \Gamma_M(\zeta),$$

for some $M > 1$. Accordingly, by saying that f is *nontangentially small of order $> \gamma$ at ζ* we mean that

$$f(z) = o((1 - |z|)^\gamma) \quad \text{as } |z| \rightarrow 1, \ z \in \Gamma_M(\zeta),$$

for some $M > 1$. Finally, if f_0, \dots, f_n are linearly independent functions in $\mathcal{H}(\mathbb{D})$, then we denote by $E_\gamma(f_0, \dots, f_n)$ the set of all points $\zeta \in \mathbb{T}$ with the following property: there exists a nontrivial linear combination $\sum_{j=0}^n \lambda_j f_j$ which is nontangentially small of order $> \gamma$ at ζ (the coefficients λ_j will of course depend on ζ).

Theorem 2.1. *Let f_0, \dots, f_n be linearly independent functions in $A^{-\beta}$ (with $\beta > 0$), and put*

$$(2.3) \quad \gamma = n\beta + \frac{n(n+1)}{2}.$$

Then $E_\gamma(f_0, \dots, f_n)$ is a set of Lebesgue measure 0 on \mathbb{T} . The same is true for $\beta = 0$, provided that $A^{-\beta}$ is replaced by H^∞ , the space of bounded analytic functions.

We do not know whether the value given by (2.3) is optimal. Note, however, that the result breaks down for all $\gamma < n$. Indeed, for any fixed $\zeta \in \mathbb{T}$, the function $z \mapsto (z - \zeta)^n$ is a linear combination of $1, z, \dots, z^n$ and is nontangentially small of

order n at ζ . Thus, $E_\gamma(1, z, \dots, z^n) = \mathbb{T}$ whenever $\gamma < n$. This shows, in particular, that (2.3) is optimal for $\beta = 0$ and $n = 1$.

Before proceeding with the proof of Theorem 2.1, we state and prove a preliminary result to rely upon. Below, we write \mathbb{N} for the set of positive integers, and we use the notation $B(z, r)$ for the disk $\{w : |w - z| < r\}$.

Lemma 2.2. *Let $\alpha \in \mathbb{R}$, $\delta \in (0, 1)$ and $k \in \mathbb{N}$. Further, let G and G_0 be subsets of \mathbb{D} such that*

$$(2.4) \quad \bigcup_{z \in G} B(z, \delta(1 - |z|)) \subset G_0.$$

Then, for every function $f \in \mathcal{H}(\mathbb{D})$ satisfying

$$(2.5) \quad f(z) = O((1 - |z|)^\alpha), \quad z \in G_0,$$

we have

$$(2.6) \quad f^{(k)}(z) = O((1 - |z|)^{\alpha-k}), \quad z \in G.$$

The constant in the latter O -condition depends on that in (2.5), as well as on α , δ and k .

Proof. Fix $z \in G$ and consider the circle

$$\gamma_z = \gamma_{z, \delta} := \{\zeta \in \mathbb{C} : |\zeta - z| = \delta(1 - |z|)\}.$$

We have then

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\gamma_z} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta,$$

whence

$$(2.7) \quad |f^{(k)}(z)| \leq \frac{k!}{\delta^k(1 - |z|)^k} \cdot \sup\{|f(\zeta)| : \zeta \in \gamma_z\}.$$

It follows from (2.4) that $\gamma_z \subset \text{clos } G_0$, and so (2.5) yields

$$|f(\zeta)| \leq C(1 - |\zeta|)^\alpha, \quad \zeta \in \gamma_z,$$

with a constant $C > 0$. We now combine this with the inequalities

$$1 - \delta \leq \frac{1 - |\zeta|}{1 - |z|} \leq 1 + \delta, \quad \zeta \in \gamma_z,$$

to get

$$\sup\{|f(\zeta)| : \zeta \in \gamma_z\} \leq \tilde{C}(1 - |z|)^\alpha, \quad z \in G,$$

with a suitable $\tilde{C} = \tilde{C}(\alpha, \delta, C)$. Plugging this last estimate into (2.7), we arrive at (2.6). \square

We also need the “little oh” version of the above lemma, which can be established in a similar way.

Lemma 2.3. *Assume, under the hypotheses of Lemma 2.2, that $\mathbb{T} \cap \text{clos } G \neq \emptyset$. Then, given a function $f \in \mathcal{H}(\mathbb{D})$ satisfying*

$$f(z) = o((1 - |z|)^\alpha) \quad \text{as } |z| \rightarrow 1, \quad z \in G_0,$$

it follows that

$$f^{(k)}(z) = o((1 - |z|)^{\alpha-k}) \quad \text{as } |z| \rightarrow 1, \quad z \in G.$$

Proof of Theorem 2.1. Let $\zeta \in E_\gamma(f_0, \dots, f_n)$, so that there exist coefficients $\lambda_j = \lambda_j(\zeta)$ with $\sum_{j=0}^n |\lambda_j| > 0$ which make the linear combination

$$(2.8) \quad \sum_{j=0}^n \lambda_j f_j =: g,$$

nontangentially small of order $> \gamma$ at ζ . At least one of the coefficients, say λ_k , is thus nonzero. Recalling the notation (1.2) for the Wronskian, we put

$$W := W(f_0, \dots, f_n)$$

and

$$W_k := W(f_0, \dots, f_{k-1}, g, f_{k+1}, \dots, f_n).$$

It should be noted that $W \not\equiv 0$, because the f_j 's are linearly independent holomorphic functions. Furthermore, it follows from (2.8) that $W_k = \lambda_k W$.

Expanding the determinant W_k along its k th column (i.e., the one that contains $g, g', \dots, g^{(n)}$), we get

$$(2.9) \quad W_k = \sum_{l=0}^n g^{(l)} \Delta_l,$$

where $\Delta_l = \Delta_{l,k}$ are the corresponding cofactors. We now claim that each term in this sum has nontangential limit 0 at ζ . Consider, for example, the last term $g^{(n)} \Delta_n$. First of all, since g is nontangentially small of order $> \gamma$ at ζ , Lemma 2.3 shows that $g^{(n)}$ is nontangentially small of order $> \gamma - n$ at ζ ; that is,

$$(2.10) \quad g^{(n)}(z) = o((1 - |z|)^{\gamma-n}) \quad \text{as } |z| \rightarrow 1, \quad z \in \Gamma_{\widetilde{M}}(\zeta),$$

for some $\widetilde{M} > 1$. (We have applied the lemma with $G_0 = \Gamma_M(\zeta)$ and $G = \Gamma_{\widetilde{M}}(\zeta)$, where $1 < \widetilde{M} < M$. The hypothesis (2.4) is then fulfilled with a suitable $\delta = \delta(M, \widetilde{M})$.) As to the cofactor

$$\Delta_n = (-1)^{k+n} \det \left\{ f_j^{(s)} : 0 \leq j \leq n \ (j \neq k), \ 0 \leq s \leq n-1 \right\},$$

it is the sum of $n!$ products of the form

$$(2.11) \quad \pm f_{j_1} f'_{j_2} \dots f_{j_n}^{(n-1)},$$

where the multiindex (j_1, \dots, j_n) runs through the permutations of

$$(0, \dots, k-1, k+1, \dots, n).$$

For all j and s , we have $f_j \in A^{-\beta}$ and hence $f_j^{(s)} \in A^{-\beta-s}$ (apply Lemma 2.2 with $G = G_0 = \mathbb{D}$, $\delta = \frac{1}{2}$ and $\alpha = -\beta$), so that

$$f_j^{(s)}(z) = O((1 - |z|)^{-\beta-s}), \quad z \in \mathbb{D}.$$

It follows that the products (2.11) are all $O((1 - |z|)^{-\kappa})$, where

$$\kappa = n\beta + 1 + 2 + \cdots + (n-1) = n\beta + \frac{n(n+1)}{2} - n = \gamma - n.$$

A similar estimate therefore holds for Δ_n ; thus

$$\Delta_n(z) = O((1 - |z|)^{n-\gamma}), \quad z \in \mathbb{D}.$$

Combining this with (2.10), we obtain

$$g^{(n)}(z)\Delta_n(z) \rightarrow 0 \quad \text{as } |z| \rightarrow 1, \quad z \in \Gamma_{\widetilde{M}}(\zeta).$$

The other terms on the right-hand side of (2.9) are treated similarly, and we conclude that W_k has nontangential limit 0 at ζ . The same is then true for $W = W_k/\lambda_k$, so we see that $E_\gamma(f_0, \dots, f_n)$ is a subset of

$$\{\zeta \in \mathbb{T} : W(z) \rightarrow 0 \text{ as } z \rightarrow \zeta \text{ nontangentially}\}.$$

This last set has Lebesgue measure 0 by virtue of the Lusin–Privalov uniqueness theorem (see [10]), and the required result follows. \square

Remark. In the above proof, Lemma 2.3 was applied to the case where G and G_0 are two Stolz angles with the same vertex. Other choices of G and G_0 , with (2.4) fulfilled, may lead to further variations on the theme of Theorem 2.1. One such choice can be described as follows: given a function $h \in H^\infty$ with $\|h\|_\infty = 1$, fix a number $\varepsilon \in (0, 1)$ and take the level set

$$\Omega(h, \varepsilon) := \{z \in \mathbb{D} : |h(z)| < \varepsilon\}$$

as G_0 , then put $G = \Omega(h, \varepsilon/2)$. The situation becomes nontrivial if the closure of G hits \mathbb{T} . We shall return to this in the next section; in particular, see Lemma 3.3 below.

3. SMOOTH ANALYTIC FUNCTIONS AND INNER FACTORS

By a “smooth analytic function” we mean a function in $\mathcal{H}(\mathbb{D})$ that is smooth up to \mathbb{T} . Specifically, we are concerned here with the analytic Lipschitz–Zygmund spaces A^α for $\alpha > 0$. Recall that a function $f \in \mathcal{H}(\mathbb{D})$ is said to be in A^α if it obeys condition (2.2), where m is the smallest integer in (α, ∞) . In fact, taking m to be any other integer in that interval, one arrives at an equivalent definition. It should be mentioned that, in the case $\alpha \in (0, \infty) \setminus \mathbb{N}$, an analytic function f will be in A^α if and only if there exists a constant $C = C_f$ such that

$$(3.1) \quad |f^{(k)}(z) - f^{(k)}(w)| \leq C|z - w|^{\alpha-k}, \quad z, w \in \mathbb{D},$$

where $k = [\alpha]$ is the integral part of α ; this is a classical result of Hardy and Littlewood. The space A^1 , known as the analytic *Zygmund class*, can be described by the appropriate second order difference condition on f . The higher order Zygmund classes A^k with $k = 2, 3, \dots$ are related to it by the formula $A^k = \{f : f^{(k-1)} \in A^1\}$.

We further recall that a closed subset E of $\mathbb{D} \cup \mathbb{T}$ will be the zero set of some non-null function in A^α , with any fixed $\alpha > 0$, or in $A^\infty := \bigcap_{0 < \alpha < \infty} A^\alpha$ if and only if it has the two properties below: first,

$$(3.2) \quad \sum_{z \in E \cap \mathbb{D}} (1 - |z|) < \infty$$

(i.e., $E \cap \mathbb{D}$ satisfies the *Blaschke condition*), and second,

$$(3.3) \quad \int_{\mathbb{T}} \log \text{dist}(\zeta, E) |d\zeta| > -\infty$$

(which is known as the *Carleson condition*). This characterization is due to Carleson [3] in the case where $E \subset \mathbb{T}$, with α finite, and to Taylor and Williams [33] in the general case.

In what follows, a subset E (not necessarily closed) of $\mathbb{D} \cup \mathbb{T}$ will be called a *(BC)-set* if it satisfies (3.2) and (3.3). In other words, E is a (BC)-set if and only if its closure, $\text{clos } E$, is a zero set for A^α or A^∞ . The closed (BC)-sets that are contained in \mathbb{T} are called *Carleson sets*; these are described by condition (3.3) alone. Of course, such sets have Lebesgue measure 0 on \mathbb{T} .

Now suppose that $n \in \mathbb{N}$ and $f \in A^\alpha$ with $\alpha > n$. Then $f^{(n)}$ is continuous up to \mathbb{T} , and there is an obvious way to extend the notion of an n -deep zero to a boundary point: just say that f has an n -deep zero at a point $\zeta \in \mathbb{T}$ if $f^{(l)}(\zeta) = 0$ for $l = 0, \dots, n$. This done, we arrive at the following version of Theorem A for A^α functions.

Theorem 3.1. *Suppose f_0, \dots, f_n are linearly independent functions in A^α , where $\alpha > n$. Then there is a (BC)-set \mathcal{E} with the following property: whenever $\lambda_0, \dots, \lambda_n$ are complex numbers with $\sum_{j=0}^n |\lambda_j| > 0$, the n -deep zeros of the function $\sum_{j=0}^n \lambda_j f_j$ in $\mathbb{D} \cup \mathbb{T}$ are all contained in \mathcal{E} .*

The proof is essentially the same as that of Theorem A. The only new feature is that the exceptional set \mathcal{E} , defined again as the zero set of $W := W(f_0, \dots, f_n)$, will now be a (BC)-set. This is due to the fact that, under the current assumptions, W is a nontrivial function in $A^{\alpha-n}$.

Our next result, Theorem 3.2 below, is similar in nature but deals with a somewhat more sophisticated situation. This time, the boundary smallness condition imposed on $g = \sum_{j=0}^n \lambda_j f_j$ will be expressed by saying that g multiplies (every power of) a certain inner function into A^α . Recall that a function $\theta \in H^\infty$ is said to be *inner* if

$$|\theta(\zeta)| = \lim_{r \rightarrow 1^-} |\theta(r\zeta)| = 1$$

at almost all points $\zeta \in \mathbb{T}$. Further, given $f \in A^\alpha$ and an inner function θ , we say that f is *strongly multipliable by θ in A^α* to mean that

$$(3.4) \quad f\theta^k \in A^\alpha \quad \text{for all } k \in \mathbb{N}.$$

When $0 < \alpha < 1$, (3.4) is equivalent to the (formally) weaker condition that $f\theta \in A^\alpha$, but for larger α 's this last condition becomes actually weaker in general; see [31, 32] and [16] for a discussion of this phenomenon.

The following theorem, to be found in [12, 13], provides a criterion for a function $f \in A^\alpha$ to be strongly multipliable by an inner function θ in A^α . See also [14, 15, 22] for alternative versions and approaches. The criterion will be stated in terms of a decrease condition to be satisfied by f along the set

$$\Omega(\theta, \varepsilon) := \{z \in \mathbb{D} : |\theta(z)| < \varepsilon\}, \quad 0 < \varepsilon < 1.$$

Theorem B. *Let $0 < \alpha < \infty$ and let m be an integer with $m > \alpha$. Given $f \in A^\alpha$ and an inner function θ , the following conditions are equivalent.*

(i.B) $f\theta^m \in A^\alpha$.

(ii.B) f is strongly multipliable by θ in A^α .

(iii.B) For some $\varepsilon \in (0, 1)$, one has

$$f(z) = O((1 - |z|)^\alpha), \quad z \in \Omega(\theta, \varepsilon).$$

Yet another equivalent condition is obtained from (iii.B) upon replacing the word “some” by “each”; see [12] or [13]. The constant in the O -condition will, of course, depend on ε . Finally, we remark that if any of the conditions (i.B)–(iii.B) holds for an $f \in A^\alpha$, $f \not\equiv 0$, with a nontrivial inner function θ (where “nontrivial” means distinct from a finite Blaschke product), then f vanishes on

$$(3.5) \quad \sigma(\theta) := \mathbb{T} \cap \text{clos } \Omega\left(\theta, \frac{1}{2}\right),$$

and the latter is therefore a Carleson set. This set $\sigma(\theta)$ is called the *boundary spectrum* of θ ; the value $\frac{1}{2}$ in (3.5) may safely be replaced by any other $\varepsilon \in (0, 1)$.

The main result of this section is as follows.

Theorem 3.2. *Let f_0, \dots, f_n be linearly independent functions in A^α , where $\alpha > n$. If g is a nontrivial linear combination of the f_j ’s, and if θ is an inner function such that g is strongly multipliable by θ in A^α , then $W := W(f_0, \dots, f_n)$ is strongly multipliable by θ in $A^{\alpha-n}$.*

Thus, the inner functions θ that arise in connection with *all* the nontrivial g ’s as above (in the sense that g is strongly multipliable by θ in A^α) are actually related in a similar way, but with $\alpha - n$ in place of α , to a *single* function W in $A^{\alpha-n}$. In particular, *the boundary spectrum $\sigma(\theta)$ of any such θ is contained in a fixed Carleson set* (namely, in the boundary zero set of W). This last property can no longer be guaranteed if we modify the hypotheses of Theorem 3.2 by taking $\alpha = n$, as the following example shows.

Example. Let

$$B_1(z) := \prod_{j=1}^{\infty} \frac{1 - 2^{-j} - z}{1 - (1 - 2^{-j})z}, \quad z \in \mathbb{D},$$

so that B_1 is the Blaschke product with zeros $\{1 - 2^{-j}\}$. It is fairly easy to check that, for $0 < \varepsilon < 1$, the level set $\Omega(B_1, \varepsilon)$ is contained in some Stolz angle $\Gamma_M(1)$. Consequently, given $\zeta \in \mathbb{T}$, the Blaschke product $B_\zeta(z) := B_1(\bar{\zeta}z)$ will have its level set $\Omega(B_\zeta, \varepsilon)$ in $\Gamma_M(\zeta)$. The function $g_\zeta(z) := (z - \zeta)^n$ is a linear combination of $1, \dots, z^n$ and is $O((1 - |z|)^n)$ on $\Gamma_M(\zeta)$, so we deduce from Theorem B that g_ζ is

strongly multipliable by B_ζ in A^n . On the other hand, we have $\sigma(B_\zeta) = \{\zeta\}$ and hence $\bigcup_{\zeta \in \mathbb{T}} \sigma(B_\zeta) = \mathbb{T}$.

To prove Theorem 3.2, the following elementary lemma will be needed.

Lemma 3.3. *Let $h \in H^\infty$, $\|h\|_\infty \leq 1$, and let $0 < \varepsilon < 1$. Then, for every $z \in \Omega(h, \frac{\varepsilon}{2})$, the disk $B(z, \frac{\varepsilon}{4}(1 - |z|))$ is contained in $\Omega(h, \varepsilon)$.*

Proof. One readily checks that $B(z, \frac{\varepsilon}{4}(1 - |z|))$ is contained in the non-Euclidean disk

$$K\left(z, \frac{\varepsilon}{4}\right) := \left\{w \in \mathbb{D} : \rho(z, w) < \frac{\varepsilon}{4}\right\},$$

where $\rho(\cdot, \cdot)$ is the pseudohyperbolic distance on \mathbb{D} given by

$$\rho(z, w) := \left| \frac{z - w}{1 - \bar{w}z} \right|.$$

Now, given $z \in \Omega(h, \frac{\varepsilon}{2})$ and $w \in K(z, \frac{\varepsilon}{4})$, we use the well-known inequality

$$\rho(h(z), h(w)) \leq \rho(z, w)$$

(see [23, Chapter I]) to deduce that

$$|h(z) - h(w)| \leq 2\rho(z, w) < \frac{\varepsilon}{2}$$

and hence

$$|h(w)| \leq |h(z)| + \frac{\varepsilon}{2} < \varepsilon.$$

This shows that $K(z, \frac{\varepsilon}{4}) \subset \Omega(h, \varepsilon)$, and the required conclusion follows. \square

Proof of Theorem 3.2. Let θ be inner, and suppose

$$(3.6) \quad g = \sum_{j=0}^n \lambda_j f_j$$

is a non-null function that is strongly multipliable by θ in A^α . As in the proof of Theorem 2.1 above, we now fix an index $k \in \{0, \dots, n\}$ with $\lambda_k \neq 0$ and consider the Wronskian

$$(3.7) \quad W_k := W(f_0, \dots, f_{k-1}, g, f_{k+1}, \dots, f_n),$$

so that $W_k = \lambda_k W$. We also recall that $W \not\equiv 0$ and $W \in A^{\alpha-n}$; the latter is due to the fact that all the entries of the corresponding Wronskian matrix are in $A^{\alpha-n}$.

Expanding, as before, the determinant (3.7) along the column $(g, g', \dots, g^{(n)})^T$, we obtain

$$(3.8) \quad W_k = \sum_{l=0}^n g^{(l)} \Delta_l,$$

where Δ_l are the appropriate cofactors. Since g is strongly multipliable by θ in A^α , it follows from Theorem B that

$$g(z) = O((1 - |z|)^\alpha), \quad z \in \Omega(\theta, \varepsilon),$$

for some $\varepsilon \in (0, 1)$. In view of Lemma 2.2, this yields

$$g^{(l)}(z) = O\left((1 - |z|)^{\alpha-l}\right), \quad z \in \Omega\left(\theta, \frac{\varepsilon}{2}\right),$$

for each $l \in \mathbb{N}$. (We have applied Lemma 2.2 with $G_0 = \Omega(\theta, \varepsilon)$ and $G = \Omega(\theta, \varepsilon/2)$. The hypothesis (2.4) is then fulfilled with $\delta = \varepsilon/4$, as Lemma 3.3 shows.) In particular,

$$(3.9) \quad g^{(l)}(z) = O\left((1 - |z|)^{\alpha-n}\right), \quad z \in \Omega\left(\theta, \frac{\varepsilon}{2}\right), \quad 0 \leq l \leq n.$$

The cofactors Δ_l are all in $A^{\alpha-n}$, and hence in H^∞ , so (3.8) and (3.9) together imply that

$$W_k(z) = O\left((1 - |z|)^{\alpha-n}\right), \quad z \in \Omega\left(\theta, \frac{\varepsilon}{2}\right).$$

A similar estimate holds then for $W = W_k/\lambda_k$, and another application of Theorem B convinces us that W is strongly multipliable by θ in $A^{\alpha-n}$, as desired. \square

We supplement Theorem 3.2 with the next result, which involves the *star-invariant subspace*

$$(3.10) \quad K_\theta := H^2 \ominus \theta H^2$$

of the Hardy space H^2 . Here, the term “star-invariant” means invariant under the backward shift operator $f \mapsto (f - f(0))/z$. It is well known that the (closed and proper) star-invariant subspaces of H^2 are precisely those of the form (3.10), with θ an inner function; see [28]. Also, the Korenblum spaces $A^{-\beta}$ from the previous section will now reappear, along with the Lipschitz–Zygmund A^α spaces.

Theorem 3.4. *Let f_0, \dots, f_n be linearly independent functions in A^α , where $\alpha > n$. Further, let $F \in A^{-\beta}$ with $\beta = \alpha - n$, and let θ be an inner function such that $FW \in K_\theta$, where $W := W(f_0, \dots, f_n)$. Assume finally that there exists a nontrivial linear combination of the f_j ’s which is strongly multipliable by θ in A^α . Then $FW \in H^\infty$.*

The statement becomes especially transparent when $n = 0$, in which case it reduces to the following.

Corollary 3.5. *Given $0 < \alpha < \infty$, suppose that $F \in A^{-\alpha}$, $g \in A^\alpha$, and θ is an inner function. If g is strongly multipliable by θ in A^α , and if $Fg \in K_\theta$, then $Fg \in H^\infty$.*

We now cite, as Lemma 3.6 below, a remarkable “maximum principle” for K_θ that was proved by Cohn in [7].

Lemma 3.6. *Let θ be inner, and let $f \in K_\theta$. If*

$$(3.11) \quad \sup\{|f(z)| : z \in \Omega(\theta, \varepsilon)\} < \infty$$

for some $\varepsilon \in (0, 1)$, then $f \in H^\infty$.

Proof of Theorem 3.4. Proceeding as in the proof of Theorem 3.2, we verify that $W := W(f_0, \dots, f_n)$ is strongly multipliable by θ in A^β , so that

$$W(z) = O\left((1 - |z|)^\beta\right), \quad z \in \Omega(\theta, \varepsilon),$$

with a suitable $\varepsilon \in (0, 1)$. Since $F \in A^{-\beta}$, we also have

$$F(z) = O\left((1 - |z|)^{-\beta}\right), \quad z \in \mathbb{D}.$$

Consequently, the function $f := FW$ satisfies (3.11), and Lemma 3.6 now tells us that $f \in H^\infty$. \square

4. HARDY–SOBOLEV SPACES: DEEP ZEROS AND SINGULAR FACTORS

Here, we shall be concerned with functions in the *Hardy–Sobolev spaces*

$$H_k^p := \{f \in \mathcal{H}(\mathbb{D}) : f^{(k)} \in H^p\},$$

with $p > 0$ and $k \in \mathbb{N}$, where $H^p = H^p(\mathbb{D})$ are the usual (holomorphic) *Hardy spaces* on the disk.

It is well known that H_1^p is contained in the Lipschitz space $A^{1-1/p}$ for $p > 1$, while

$$(4.1) \quad H_1^1 \subset H^\infty$$

(moreover, the functions from H_1^1 are continuous up to \mathbb{T}) and

$$(4.2) \quad H_1^p \subset H^{p/(1-p)} \quad \text{for } 0 < p < 1.$$

These results, which are chiefly due to Hardy and Littlewood, can be found in [11]. Iterating them, one arrives at the appropriate embedding theorems for H_k^p with $k \geq 2$. In particular, we always have $H_k^p \subset H^p$.

Now let us recall that any nontrivial function $f \in H^p$ can be factored canonically as $f = I\mathcal{O}$, where I is inner and \mathcal{O} is outer. (A function $\mathcal{O} \in \mathcal{H}(\mathbb{D})$ is called *outer* if $\log |\mathcal{O}(z)|$ agrees, for $z \in \mathbb{D}$, with the harmonic extension of an integrable function on \mathbb{T} .) The inner factor I can be further decomposed as $I = BS$, where B is a *Blaschke product* and S is a *singular inner function*; see [23, Chapter II]. More explicitly, the factors B and S in this last formula are of the form

$$B(z) = B_{\{z_j\}}(z) := \prod_j \frac{\bar{z}_j}{|z_j|} \frac{z_j - z}{1 - \bar{z}_j z},$$

where $\{z_j\} \subset \mathbb{D}$ is a sequence (possibly finite or empty) with $\sum_j (1 - |z_j|) < \infty$, and

$$S(z) = S_\mu(z) := \exp \left\{ - \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \right\},$$

where μ is a (nonnegative) singular measure on \mathbb{T} .

We further remark that if the functions f_0, \dots, f_n from Theorem A are taken to be in H_n^p , then their Wronskian W is in H^q with a suitable q ; in case $p \geq 1$, this is true with $q = p$. (To verify these claims, use (4.1) and (4.2).) The zero set $\mathcal{E} = W^{-1}(0)$ therefore satisfies the Blaschke condition $\sum_{z \in \mathcal{E}} (1 - |z|) < \infty$, and we may rephrase Theorem A as saying that there exists a Blaschke product (the one built from \mathcal{E}) with a certain divisibility property. Our current purpose is to extend this to singular inner factors; in a sense, such factors can be thought of as responsible for the function's “boundary zeros of infinite multiplicity”.

A unified statement, involving both Blaschke products and singular factors, is given in Theorem 4.1 below. In particular, it turns out that if the linearly independent functions f_0, \dots, f_n are in H_n^1 , then (much in the spirit of Theorem A) there is a *single* singular inner function S divisible by the singular factor of each nontrivial linear combination $\sum_{j=0}^n \lambda_j f_j$. This means that the totality of singular factors resulting from such linear combinations is rather poor. On the other hand, we shall see that the hypothesis $f_j \in H_n^1$, or at least some smoothness assumption on the f_j 's, is indispensable; in fact, it is not enough to assume that the functions are merely in H^∞ .

Let $I = BS$ be an inner function, where B is a Blaschke product and S is singular, and let $n \in \mathbb{N}$. We write $B_{>n}$ for the Blaschke product obtained from B by removing the zeros of multiplicity $\leq n$ (the remaining zeros, if any, are kept with their multiplicities unchanged); then we put $I_{>n} = B_{>n}S$.

Theorem 4.1. *Let f_0, \dots, f_n be linearly independent functions in H_n^1 . Then there is an inner function J with the following property: whenever $\lambda_0, \dots, \lambda_n$ are complex numbers with $\sum_{j=0}^n |\lambda_j| > 0$ and I is the inner factor of $\sum_{j=0}^n \lambda_j f_j$, the inner function $I_{>n}$ divides J .*

Proof. Because the f_j 's are linearly independent, the Wronskian

$$W := W(f_0, \dots, f_n)$$

is non-null. In addition, $W \in H^1$. To verify this last fact, expand the determinant (1.2) along its last row and observe that $f_j^{(n)} \in H^1$ for each j , while the derivatives $f_j^{(\nu)}$ with $0 \leq \nu \leq n-1$ are all in H^∞ . Consider the inner factor $B_W S_W$ of W ; here B_W is a Blaschke product and S_W is singular. Further, let z_l ($l = 1, 2, \dots$) be the distinct zeros of B_W , of respective multiplicities m_l . We now form a new Blaschke product \tilde{B} with the same zero set $\{z_l\}$, this time assigning multiplicity $m_l + n$ to z_l . Our plan is to check that the inner function $J := \tilde{B}S_W$ has the required property.

Let

$$(4.3) \quad g = \sum_{j=0}^n \lambda_j f_j$$

be a nontrivial linear combination of the f_j 's, and let $I = I_g = B_g S_g$ be the inner factor of g . (Again, it is understood that B_g is a Blaschke product and S_g is singular.) Next, we fix an index $k \in \{0, \dots, n\}$ for which the coefficient λ_k in (4.3) is nonzero, and we put

$$W_k := W(f_0, \dots, f_{k-1}, g, f_{k+1}, \dots, f_n).$$

Thus, W_k is the determinant obtained from (1.2) by replacing its k th column with

$$(4.4) \quad (g, g', \dots, g^{(n)})^T,$$

and it follows from (4.3) that $W_k = \lambda_k W$. In particular, the inner factors of W and W_k are identical.

Now assume that g has a zero of multiplicity μ , $\mu > n$, at a point $\zeta \in \mathbb{D}$ (so that ζ is a zero of $I_{>n}$). The derivative $g^{(\nu)}$, with $\nu = 1, \dots, n$, will then vanish to order

$\mu - \nu$ at ζ . Therefore, expanding the determinant W_k along its k th column (4.4), we see that W_k (and hence W , as well as B_W) has a zero of multiplicity at least $\mu - n$ at ζ . Consequently, ζ coincides with z_l for some l , and $\mu - n \leq m_l$. Thus μ does not exceed $m_l + n$ (which is the multiplicity of z_l as a zero of \tilde{B}), and we deduce that \tilde{B} is divisible by b_ζ^μ , where $b_\zeta(z) = \frac{\zeta - z}{1 - \bar{\zeta}z}$. This shows that the Blaschke factor of $I_{>n}$ divides \tilde{B} , the Blaschke factor of J .

Finally, we need to deal with the singular parts of $I_{>n}$ and J . Specifically, we must check that S_g divides S_W . To this end, we notice that the inner factors of $g', g'', \dots, g^{(n)}$ are all divisible by S_g . (In fact, it is known that for every $h \in H_1^1$, the singular factor of h divides that of h' ; see [4] or [34].) Once again, we expand the determinant W_k along its k th column (4.4), while noting that the corresponding cofactors are in H^1 , to conclude that the singular inner factor of W_k (i.e., S_W) is indeed divisible by S_g . The proof is now complete. \square

When specialized to singular factors, Theorem 4.1 takes a simpler form.

Theorem 4.2. *Let f_0, \dots, f_n be linearly independent functions in H_n^1 . Then there is a singular inner function S with the following property: whenever $\lambda_0, \dots, \lambda_n$ are complex numbers with $\sum_{j=0}^n |\lambda_j| > 0$, the singular factor of $\sum_{j=0}^n \lambda_j f_j$ divides S .*

From this, yet another fact will be deduced. But first we cite, as Lemma 4.3 below, a somewhat restricted version of a result from [36]. With a further application in mind, we state it for a generic domain $\Omega \subset \mathbb{C}$ rather than for the disk.

Lemma 4.3. *If $f \in \mathcal{H}(\Omega)$, then*

$$W(1, f, f^2, \dots, f^n) = c_n [f']^{n(n+1)/2},$$

where $c_n = \prod_{k=1}^n k!$.

To derive the next corollary, it suffices to apply Theorem 4.2 to the case where the f_j 's are powers of a single function. In doing so, one should take the function S in that theorem to be the singular factor of the Wronskian (in accordance with the preceding proof) and combine this with Lemma 4.3.

Corollary 4.4. *Let f be a nonconstant function in H_n^1 , and let \mathbf{S} be the singular inner factor of f' . Then $\mathbf{S}^{n(n+1)/2}$ is divisible by the singular inner factor of every linear combination $\sum_{k=0}^n \lambda_k f^k$ with $\sum_{k=0}^n |\lambda_k| > 0$. In particular, if f' has no singular factor, then the same is true for each of the linear combinations in question.*

Finally, we show that Theorem 4.2 (and hence also Theorem 4.1) becomes false, already for $n = 1$, if we replace H_n^1 by H^∞ .

Proposition 4.5. *There are functions $f_0, f_1 \in H^\infty$ with the following property: for each singular inner function S there is a nontrivial linear combination $\lambda_0 f_0 + \lambda_1 f_1$ whose singular factor does not divide S .*

Proof. We borrow an idea from [5]. Let θ be a nonconstant inner function that omits an uncountable set of values $\mathcal{A} \subset \mathbb{D}$. (The existence of such a function with values

in $\mathbb{D} \setminus \mathcal{A}$, for any prescribed closed set \mathcal{A} of zero logarithmic capacity, is established in [10, Chapter 2].) For each $\alpha \in \mathcal{A}$, one has

$$(4.5) \quad \theta - \alpha = S_\alpha \cdot (1 - \bar{\alpha}\theta),$$

with

$$S_\alpha := \frac{\theta - \alpha}{1 - \bar{\alpha}\theta}.$$

Here, S_α is a singular inner function (because α is not in the range of θ), while the other factor in (4.5) is outer.

Write μ_α for the singular measure associated with S_α . For μ_α -almost every $\zeta \in \mathbb{T}$, we have $S_\alpha(z) \rightarrow 0$, and hence $\theta(z) \rightarrow \alpha$, as $z \rightarrow \zeta$ nontangentially; see [23, Chapter II]. It follows that the supports of μ_α 's, with $\alpha \in \mathcal{A}$, are pairwise disjoint. The set \mathcal{A} being uncountable and the measures μ_α nonzero, we readily deduce that no finite Borel measure μ on \mathbb{T} can satisfy $\mu \geq \mu_\alpha$ for all $\alpha \in \mathcal{A}$. Consequently, no singular inner function is divisible by every S_α .

This said, we put $f_0 := \theta$ and $f_1 := 1$. Since S_α is the singular factor of $\theta - \alpha$, which is a linear combination of f_0 and f_1 , we are done. \square

It would be interesting to know if the space H_n^1 in Theorems 4.1 and 4.2 can be replaced by a larger smoothness class (say, by H_n^p with a $p < 1$), and moreover, to find the optimal smoothness conditions on the functions f_j that guarantee the validity of those results.

5. ZERO SETS OF WRONSKIANS

In this section, we discuss the exceptional sets \mathcal{E} that may arise in Theorem A (see Introduction) when the functions f_0, \dots, f_n are assumed to lie in a specific space X of holomorphic functions on a domain $\Omega \subset \mathbb{C}$. For the time being, we prefer to deal with a general domain, not necessarily with the disk. Given a space $X \subset \mathcal{H}(\Omega)$, we now introduce the appropriate concepts and notations.

Definition. (a) Let f_0, \dots, f_n be linearly independent functions in $\mathcal{H}(\Omega)$. We write $\mathcal{Z}(f_0, \dots, f_n)$ for the set of points $z \in \Omega$ with the following property: there exist complex numbers $\lambda_0 = \lambda_0(z), \dots, \lambda_n = \lambda_n(z)$ with $\sum_{j=0}^n |\lambda_j| > 0$ such that the function $\sum_{j=0}^n \lambda_j f_j$ has an n -deep zero at z .

(b) We denote by $\mathcal{Z}_n(X)$ the collection of those sets $E \subset \Omega$ which can be written as $E = \mathcal{Z}(f_0, \dots, f_n)$ for some linearly independent functions f_0, \dots, f_n in X .

The proof of Theorem A from the Introduction shows that $\mathcal{Z}(f_0, \dots, f_n)$ is no other than the zero set of $W(f_0, \dots, f_n)$. We further remark that, when $n = 0$, $\mathcal{Z}(f_0)$ is just the zero set of f_0 , while $\mathcal{Z}_0(X)$ is the class of zero sets for X .

In what follows, we write $\mathcal{Z}(X)$ for $\mathcal{Z}_0(X)$; a (discrete) set E in Ω will thus belong to $\mathcal{Z}(X)$ if and only if it coincides with $\mathcal{Z}(g) := g^{-1}(0)$ for some non-null function $g \in X$. Also, we put

$$(5.1) \quad X' := \{f' : f \in X\}.$$

Theorem 5.1. *Let $X \subset \mathcal{H}(\Omega)$ and $n \in \mathbb{N}$. Assume, in addition, that X is an algebra (with respect to the usual pointwise multiplication of functions) that contains the constant function 1 and satisfies $X' = X$. Then $\mathcal{Z}_n(X) = \mathcal{Z}(X)$.*

Proof. Let $E \in \mathcal{Z}_n(X)$, so that E is the zero set of $W := W(f_0, \dots, f_n)$ for some linearly independent functions $f_0, \dots, f_n \in X$. Our assumptions on X imply that $W \in X$ (because all the entries of the Wronskian matrix are in X), and so $E \in \mathcal{Z}(X)$. This proves the inclusion $\mathcal{Z}_n(X) \subset \mathcal{Z}(X)$.

Conversely, suppose $E \in \mathcal{Z}(X)$, so that $E = g^{-1}(0)$ for some non-null function $g \in X$. We then write $g = f'$ for a suitable $f \in X$ and invoke Lemma 4.3 to get

$$W(1, f, \dots, f^n) = c_n g^{n(n+1)/2}.$$

This last Wronskian (which is built from the functions f^k lying in X) vanishes precisely on E , so $E \in \mathcal{Z}_n(X)$; this shows that $\mathcal{Z}(X) \subset \mathcal{Z}_n(X)$. \square

As examples of algebras X satisfying the hypotheses of Theorem 5.1, with $\Omega = \mathbb{D}$, we mention $A^\infty := \bigcap_{\alpha > 0} A^\alpha$ and $A^{-\infty} := \bigcup_{\beta > 0} A^{-\beta}$. In the former case, the family $\mathcal{Z}(X)$ is formed by the closed (BC)-sets (see Section 3 above), while in the latter case it is characterized by Korenblum's density condition (see [24]). To give yet another example, this time with $\Omega = \mathbb{C}$, fix a number $\rho \in (0, \infty)$ and take X to be the space of entire functions of order at most ρ and of finite type. For a description of $\mathcal{Z}(X)$ in this last example, we refer to [26, Chapter I].

For the rest of the paper, we go back to the case $\Omega = \mathbb{D}$.

Theorem 5.2. *Let m and n be nonnegative integers, and let $0 < p < \infty$.*

(i) *In order that every set E in $\mathcal{Z}_n(H_m^p)$ satisfy the Blaschke condition*

$$\sum_{z \in E} (1 - |z|) < \infty,$$

it is necessary and sufficient that $m \geq n$.

(ii) *In order that every set in $\mathcal{Z}_n(H_m^p)$ be a (BC)-set, it is necessary and sufficient that $m \geq n + p^{-1}$.*

A few preliminary results will be needed.

Lemma 5.3. *Given $p > 0$ and $k \in \mathbb{N}$ with $kp \geq 1$, one has $H_k^p \subset H_1^1$.*

Indeed, the case $p \geq 1$ is trivial in view of (4.1), while for $0 < p < 1$ the required fact can be established by repeated application of (4.2).

Lemma 5.4. *The space H_1^1 is an algebra, and every zero set for H_1^1 is a (BC)-set.*

Here, the first statement is an easy consequence of (4.1); the second, which is much deeper, was proved in [35]. See also [6] in connection with boundary zero sets.

Lemma 5.5. *For each $l \in \mathbb{N}$ and $p > 0$, the space*

$$(H^p)^{(l)} := \{f^{(l)} : f \in H^p\}$$

contains a function whose zero set fails to satisfy the Blaschke condition.

To verify this, recall that even the Bloch space $\mathcal{B} = (A^1)'$ is known to contain functions with non-Blaschke zero sets (see [1]). Since $A^1 \subset H^\infty$, a similar conclusion holds for $(H^\infty)'$ and hence, *a fortiori*, for $(H^p)^{(l)}$.

Finally, the next result is a restricted version of [27, Theorem 1].

Lemma 5.6. *Let $k \in \mathbb{N}$ and $0 < \alpha < 1/(k+1)$. If $\{a_n\} \subset \mathbb{D}$ is a sequence such that*

$$(5.2) \quad \sum_n (1 - |a_n|)^\alpha < \infty$$

and B is the Blaschke product with zeros $\{a_n\}$, then $B \in H_k^p$ whenever $0 < p \leq (1 - \alpha)/k$.

From this we deduce the following fact.

Corollary 5.7. *Let $k \in \mathbb{N}$ and $0 < p < 1/k$. If B is a Blaschke product whose zero sequence $\{a_n\}$ satisfies $|a_n| = 1 - 2^{-n}$ ($n = 1, 2, \dots$), then $B \in H_k^p$.*

Indeed, it suffices to apply Lemma 5.6 with a suitably small α , for instance, with

$$\alpha = \min \left(1 - pk, \frac{1}{2(k+1)} \right).$$

Proof of Theorem 5.2. We begin by proving the sufficiency in (i) and (ii). Consider the Wronskian $W := W(f_0, \dots, f_n)$ of some linearly independent functions $f_0, \dots, f_n \in H_m^p$.

Now, if $m \geq n$, then $f_0^{(n)}, \dots, f_n^{(n)}$ are in H_k^p , where $k = m - n$, and hence in H^p . It follows that the lower order derivatives $f_j^{(l)}$, with $0 \leq j \leq n$ and $0 \leq l \leq n - 1$, are also (at least) in H^p . This in turn implies that W lies in a certain H^r space (recall that $\bigcup_{q>0} H^q$ is an algebra or simply put $r = p/(n+1)$ and use Hölder's inequality), so the zero set $\mathcal{Z}(W)$ satisfies the Blaschke condition.

Similarly, if $m \geq n + p^{-1}$, then the n th derivatives $f_j^{(n)}$ of all the f_j 's are in H_k^p with $k = m - n \geq p^{-1}$, and Lemma 5.3 ensures that $f_j^{(n)} \in H_1^1$. From this we deduce that the lower order derivatives $f_j^{(l)}$ are also (at least) in H_1^1 , which eventually yields $W \in H_1^1$ in view of Lemma 5.4. The same lemma tells us, then, that $\mathcal{Z}(W)$ is a (BC)-set.

Now let us turn to the necessity in (i) and (ii). Suppose that $m < n$ and put $\ell = n - m$. Further, invoke Lemma 5.5 to find a function $g \in (H^p)^{(\ell)}$ whose zero set $\mathcal{Z}(g)$ is non-Blaschke, in the sense that

$$\sum_{z \in \mathcal{Z}(g)} (1 - |z|) = \infty,$$

and let $f \in \mathcal{H}(\mathbb{D})$ be such that $f^{(n)} = g$. Since g is the ℓ th derivative of $f^{(m)}$, it follows that $f^{(m)} \in H^p$, or equivalently, $f \in H_m^p$. The elementary formula

$$(5.3) \quad W \left(1, \frac{z}{1!}, \dots, \frac{z^{n-1}}{(n-1)!}, f \right) = f^{(n)}$$

shows that this last Wronskian vanishes precisely on $\mathcal{Z}(g)$, whence we conclude that $\mathcal{Z}_n(H_m^p)$ contains non-Blaschke sets.

Finally, assume that $n \leq m < n + p^{-1}$ and put $k = m - n$. Let $\{a_j\}$ be a sequence in \mathbb{D} satisfying $|a_j| = 1 - 2^{-j}$ ($j = 1, 2, \dots$) and having the whole circle \mathbb{T} as its limit set. Further, write B for the Blaschke product with zeros $\{a_j\}$, and let $f \in \mathcal{H}(\mathbb{D})$ be such that $f^{(n)} = B$. We have then

$$f^{(m)} = f^{(n+k)} = B^{(k)} \in H^p$$

(by virtue of Corollary 5.7), whence $f \in H_m^p$. Now, if W stands for the Wronskian on the left-hand side of (5.3), with our current f plugged in, then (5.3) reduces to saying that $W = B$. Consequently, the zeros of W in \mathbb{D} are precisely the a_j 's, and these obviously fail to form a (BC)-set, since $\text{clos } \{a_j\} \supset \mathbb{T}$. The proof is complete. \square

Before stating our last theorem, we have to introduce a bit of notation. Namely, given a space $X \subset \mathcal{H}(\mathbb{D})$ and an integer $m \geq 0$, we write X_m for the set of those functions $f \in \mathcal{H}(\mathbb{D})$ which satisfy $f^{(k)} \in X$ with $k = 0, \dots, m$. Of course, if X is the Hardy space H^p , then X_m becomes H_m^p . The role of X will alternatively be played by $\text{BMOA} := H^1 \cap \text{BMO}$ (where BMO is the space of functions of *bounded mean oscillation* on \mathbb{T}), as well as by the *Nevanlinna class* \mathcal{N} and the *Smirnov class* \mathcal{N}^+ .

In connection with BMO , the reader is referred to [23, Chapter VI]. The class \mathcal{N} (resp., \mathcal{N}^+) is formed by the quotients u/v with $u, v \in H^\infty$, where v is zero-free (resp., outer) in \mathbb{D} ; see [23, Chapter II] for equivalent definitions and characterizations. To keep on the safe side, we also recall the notation (5.1), since this will be used again for some of our current X 's.

Theorem 5.8. *Let X be any of the following spaces: BMOA , H^p (with $0 < p < \infty$), \mathcal{N}^+ or \mathcal{N} . Then, for every integer $m \geq 0$, one has*

$$\mathcal{Z}_{m+1}(X_m) = \mathcal{Z}(\text{BMOA}').$$

While no explicit characterization of $\mathcal{Z}(\text{BMOA}')$ seems to be available, it can be shown that

$$\mathcal{Z}(A^{-1+\varepsilon}) \subset \mathcal{Z}(\text{BMOA}') \subset \mathcal{Z}(A^{-1})$$

for an arbitrarily small $\varepsilon > 0$. On the other hand, the zero sets for an $A^{-\beta}$ space with $\beta > 0$ are “almost described” – even though not completely describable – by the appropriate Korenblum-type density condition; see Seip's refinements in [29, 30] to Korenblum's original work from [24].

The proof of Theorem 5.8 will make use of the following result, which we prove first.

Lemma 5.9. *The set*

$$\mathcal{N} \cdot \text{BMOA}' := \{fg' : f \in \mathcal{N}, g \in \text{BMOA}\}$$

is a vector space that contains \mathcal{N}' .

Proof. We begin by recalling the (well-known) fact that the space BMOA' is invariant under multiplication by H^∞ functions. Indeed, if $g \in \text{BMOA}$ and $h \in H^\infty$, then the

measure

$$|g'(z)|^2(1 - |z|) dx dy$$

where $z = x + iy$, is a Carleson measure (see [23, Chapter VI]), and so is

$$|g'(z)|^2|h(z)|^2(1 - |z|) dx dy,$$

whence $g'h \in \text{BMOA}'$. Thus,

$$(5.4) \quad H^\infty \cdot \text{BMOA}' = \text{BMOA}'.$$

Now, to prove that $\mathcal{N} \cdot \text{BMOA}'$ is a vector space, we need to verify the linearity property

$$f_1, f_2 \in \mathcal{N} \cdot \text{BMOA}' \implies f_1 + f_2 \in \mathcal{N} \cdot \text{BMOA}'.$$

To this end, we write

$$f_j = \frac{u_j}{v_j} \cdot w'_j \quad (j = 1, 2),$$

where $u_j, v_j \in H^\infty$ and $w_j \in \text{BMOA}$, and where v_j is zero-free. Note that

$$f_1 + f_2 = \frac{1}{v_1 v_2} \cdot (u_1 v_2 w'_1 + u_2 v_1 w'_2).$$

Each of the two terms in brackets, and hence their sum, is then in BMOA' by virtue of (5.4), while the factor $1/(v_1 v_2)$ is in \mathcal{N} .

Finally, to check that $\mathcal{N}' \subset \mathcal{N} \cdot \text{BMOA}'$, take any $f \in \mathcal{N}'$ and write $f = u/v$ with suitable $u, v \in H^\infty$, the function v being zero-free. The formula

$$(5.5) \quad f' = \frac{1}{v^2} \cdot (u'v - uv')$$

now provides the sought-after factorization for f' , because $v^{-2} \in \mathcal{N}$ and $u'v - uv' \in \text{BMOA}'$, the latter being a consequence of (5.4). \square

We mention in passing that Lemma 5.9 admits an extension to higher order derivatives. In addition, similar results are available for the Smirnov class \mathcal{N}^+ . This, and more, can be found in [21]. See also [8, 9] for the corresponding factorization theorems in the H^p setting.

Proof of Theorem 5.8. Since

$$\text{BMOA} \subset H^p \subset \mathcal{N}^+ \subset \mathcal{N},$$

we clearly have

$$\mathcal{Z}_n(\text{BMOA}_m) \subset \mathcal{Z}_n(H_m^p) \subset \mathcal{Z}_n(\mathcal{N}_m^+) \subset \mathcal{Z}_n(\mathcal{N}_m)$$

for all n , and in particular for $n = m + 1$. Consequently, it suffices to show that

$$(5.6) \quad \mathcal{Z}(\text{BMOA}') \subset \mathcal{Z}_{m+1}(\text{BMOA}_m)$$

and

$$(5.7) \quad \mathcal{Z}_{m+1}(\mathcal{N}_m) \subset \mathcal{Z}(\text{BMOA}').$$

To check (5.6), assume that $E = \mathcal{Z}(g')$ for some $g \in \text{BMOA}$, and let $f \in \mathcal{H}(\mathbb{D})$ be such that $f^{(m)} = g$. Applying formula (5.3) with $n = m + 1$ yields

$$W\left(1, \frac{z}{1!}, \dots, \frac{z^m}{m!}, f\right) = g'.$$

The zero set of this last Wronskian is therefore E , and since the functions z^k and f are in BMOA_m , it follows that $E \in \mathcal{Z}_{m+1}(\text{BMOA}_m)$; this proves (5.6).

To verify (5.7), consider the Wronskian determinant

$$\mathcal{W} := W(f_0, \dots, f_{m+1})$$

built from some (any) linearly independent functions f_0, \dots, f_{m+1} in \mathcal{N}_m . Expanding the determinant along its last row, we get

$$(5.8) \quad \mathcal{W} = \sum_{j=0}^{m+1} f_j^{(m+1)} \Delta_j,$$

where Δ_j are the appropriate cofactors. Because the derivatives $f_j^{(k)}$ with $0 \leq k \leq m$ are all in \mathcal{N} , we see that the Δ_j 's are also in \mathcal{N} , whereas the functions $f_j^{(m+1)}$ are in \mathcal{N}' . By Lemma 5.9, for each $j \in \{0, \dots, m+1\}$ there are functions $\varphi_j \in \mathcal{N}$ and $\psi_j \in \text{BMOA}$ such that $f_j^{(m+1)} = \varphi_j \psi_j'$. Plugging this into (5.8) gives

$$\mathcal{W} = \sum_{j=0}^{m+1} (\varphi_j \Delta_j) \cdot \psi_j'.$$

Here, each summand on the right is in $\mathcal{N} \cdot \text{BMOA}'$, whence we infer (using Lemma 5.9 again) that $\mathcal{W} \in \mathcal{N} \cdot \text{BMOA}'$.

Consequently, we have

$$(5.9) \quad \mathcal{W} = \Phi \Psi'$$

with some $\Phi \in \mathcal{N}$ and $\Psi \in \text{BMOA}$; moreover, we take Φ to be zero-free. (To see that this is always possible, assume that (5.9) holds with $\Phi = \Phi_0 B$, where Φ_0 is zero-free and B is a Blaschke product. Then invoke (5.4) to find a function $\Psi_0 \in \text{BMOA}$ such that $\Psi_0' = B \Psi'$, and use the factorization $\mathcal{W} = \Phi_0 \Psi_0'$.) It now follows that the zero set $\mathcal{Z}(\mathcal{W})$ coincides with $\mathcal{Z}(\Psi')$ and is, therefore, contained in $\mathcal{Z}(\text{BMOA}')$. Inclusion (5.7) is thus established. \square

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